CCSF PHYC 4D Lecture Notes

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Chapter 4a
The Wavelike Properties of Particles (Part 1)

Wave-particle duality

- EM radiation satisfies

$$E = h\nu \qquad p = E/c = h\nu/c = h/\lambda$$

- de Broglie proposed in 1924 that all "particle/wave" entities obey these same relations, not just EM radiation:

$$E = h\nu = \hbar\omega \qquad p = h/\lambda = \hbar k$$

  These equations relate particle (E, p) and wave (ν, λ) properties of any particle/wave.

- Particles vs. non-particles.
  - A particle can only interact in discrete "lumps" (one electron? two electrons? never "one and a half" electrons).
  - A non-particle can interact as a continuum.
  - Example: Consider a classical water wave (of given wavelength and frequency). Since the amplitude of the wave is arbitrary, an arbitrary quantity of energy can be transferred from/to whatever the wave interacts with. It seems that water waves are not lumpy, and therefore are non-particles.

- Waves vs. non-waves.
  - A wave can undergo interference and diffraction effects. When a wave passes an object that blocks its energy, the wave will, in effect, diffract around it, filling in the shadow space with energy from the wave. If a wave passes through a double-slit, diffraction grating, or crystal structure, and is detected on the far side, then one observes an interference pattern similar to what is observed in Young's double-slit experiments or Bragg's X-ray scattering experiments.
  - A non-wave does not exhibit interference and diffraction effects.
  - Example: If a bullet-proof object were placed in front of a target that is then showered with bullets, the shadow will appear on the target as an area completely free of bullets. Bullets do not "diffract

around” bullet-proof objects — those objects completely block the bullet from reaching the shadow region. Furthermore, a “double-slit” experiment with bullets will show two piles of bullets, one on the far side of each slit, but no interference pattern.

- Before quantum mechanics, the following was assumed:

Entity	Particle?	Wave?
Water waves	No	Yes
EM radiation	No	Yes
Electrons	Yes	No
Bullets	Yes	No

- Do non-particles really exist? How can we tell the difference between “lumpy” and “non-lumpy” if the lumps are really small (i.e, E and p are both small)?

- Do non-waves really exist? How can we tell the difference between no interference/diffraction and the interference/diffraction pattern of a wave with an extremely short wavelength? For a double-slit experiment:

$$d \sin\phi = m\lambda \quad \text{(bright)} \qquad d \sin\phi = (m + \tfrac{1}{2})\lambda \quad \text{(dark)}$$

  The pattern will be unresolvable if λ is too short.

- Apply de Broglie to each example above:

$$h = 6.626 \times 10^{-34}\,\mathrm{J\,s} = 0.004136\,\mathrm{eV/THz} = 1240\,(\mathrm{eV}/c)\,\mathrm{nm}$$

Entity	λ	ν	per quanta p	E
Water waves	$10.0\,\mathrm{cm}$	$3.95\,\mathrm{Hz}$	$12.4\,\mu\mathrm{eV}/c$	$16.3\,\mathrm{feV}$
($\lambda = 10.0\,\mathrm{cm} \ll$ water depth)				
EM radiation	$500\,\mathrm{nm}$	$600\,\mathrm{THz}$	$2.48\,\mathrm{eV}/c$	$2.48\,\mathrm{eV}$
($\lambda = 500\,\mathrm{nm}$)				
Electron	$54.9\,\mathrm{pm}$	$121\,\mathrm{PHz}$	$22.6\,\mathrm{keV}/c$	$500\,\mathrm{eV}$
($K = 500\,\mathrm{eV}$)				
Bullets	$1.32 \times 10^{-34}\,\mathrm{m}$	$1.89 \times 10^{36}\,\mathrm{Hz}$	$5.00\,\mathrm{kg\,m/s}$	$1250\,\mathrm{J}$
($m = 10\,\mathrm{g}$, $v = 500\,\mathrm{m/s}$)				

- Note: the "E" above for the "traditional particles" includes only kinetic energy. There is some controversy as to whether rest energy should be included or not. The corresponding frequency is affected by this choice of zero energy, although frequency differences are not affected (relative phases of interfering waves will thus be unaffected). Still, when the wave is due to some mechanical oscillation (e.g., water waves), there is a definite, measurable frequency associated with that wave.

- Note: For deep water waves (h = water depth $\gg \lambda$):

$$v_{\text{ph}} = \sqrt{g\lambda/2\pi} \qquad \omega = \sqrt{gk}$$

  For shallow water waves ($h \ll \lambda$):

$$v_{\text{ph}} = \sqrt{gh} \qquad \omega = (\sqrt{gh})k$$

- A number of experiments followed de Broglie's hypothesis. They are outlined in the book in Section 4.2[4.1]. They essentially involve electrons, protons, neutrons, helium atoms, etc. incident upon a "double slit", "diffraction grating", or atomic crystal (as in Bragg scattering). In each case, the wavelength predicted by de Broglie's hypothesis matches closely the wavelength associated with the observed interference pattern.

Electron double-slit experiment

- Imagine $K = 500\,\text{eV}$ electrons ($\lambda = 54.9\,\text{pm}$) passing through a double-slit ($d = 1.0\,\mu\text{m}$) and continuing on to a screen $D = 5.0\,\text{m}$ away. The "bright fringes" of the interference pattern are separated by

$$\begin{aligned} \Delta y &= \Delta(D\tan\phi) \approx D\Delta(\sin\phi) = D(\Delta m)\lambda/d \\ &= (5.0\,\text{m})(1)(54.9\,\text{pm})/(1.0\,\mu\text{m}) = 0.275\,\text{mm} \end{aligned}$$

- What causes the interference pattern? What "type" of wave are we talking about? What aspect of the electrons is interfering?

- These waves are *probability* waves. Bright fringes correspond to points on the screen where the electrons are *more likely* to end up after passing through the double slit.

- A bright fringe is where constructive interference occurs: the (probability) amplitude at those points is twice as much as it would be if one slit were covered. The electron is *four* times more likely to end up at such points (within some small interval) with both slits open rather than one.

- A dark fringe is where destructive interference occurs: the (probability) amplitude at those points is close to zero. The electrons are unlikely to end up at such points with both slits open. However, with only one slit open, the amplitude is not zero. The electron is considerably more likely to end up at such points when one slit is covered than with both slits open. *This is very strange behavior — totally unexplainable from classical physics.*

- On average, a given electron emitted from a source on one side of a double-slit is twice as likely to end up somewhere on the other side of the double slit with both slits open, rather than if only one slit is open. This is expected behavior.

- This is not interference between two *different* electrons. *Every electron propogates separately as a wave, and interferes with itself.* If the double-slit experiment is carried out with an electron rate so low that only one electron occupies the apparatus at a time, the same interference pattern will still result.

- For a given electron, which slit does it pass through? It turns out that such questions cannot be answered without destroying the interference pattern. Once it is known (or knowable) which slit an electron passed through (e.g., by scattering suffiently low λ photons off of the electrons), contributions to the probability amplitude from paths passing through the other slit are eliminated.

- The double-slit experiment involving EM radiation can be interpreted the same way. Photons are emitted (in a particle-like manner) by a source, and then interfere with themselves as a probability wave when they pass through the double-slit. They strike the screen as particles, leaving a “mark” as they do so. Bright fringes occur at points where the photons are *more likely* to end up.

- Energy intensity is directly proportional to probability intensity: each photon carries an energy of $h\nu$. Doubling the photon probability doubles the photon rate, and thus doubles the energy flux per unit area (intensity) of the EM radiation.

The Heisenberg Uncertainty Principle

- One of the consequences of the wave nature of “particles” is the Heisenberg Uncertainty Principle: the fact that a particle’s position and momentum cannot be measured simultaneously to a precision better than given by the following:

  $$(\Delta p_x)(\Delta x) \geq \hbar/2 \qquad (\Delta p_y)(\Delta y) \geq \hbar/2 \qquad (\Delta p_z)(\Delta z) \geq \hbar/2$$

  Likewise, the energy of a particle cannot be measured to a precision better than given by the following:

  $$(\Delta E)(\Delta t) \geq \hbar/2$$

  where Δt is the time available for the measurement. “Δ” in this context means uncertainty.

- We will see how these uncertainty principles follow from the wave-particle duality when we return to Chapter 4 later. For now, let’s consider some consequences.

- If a particle is confined to a small region (i.e., Δx is small), then the uncertainty in the corresponding momentum component must be large: $\Delta p_x \geq \hbar/(2\Delta x)$.

- Likewise, if a particle momentum in a certain direction has a very small uncertainty, then the particle itself must be spread over a large region ($\Delta x \geq \hbar/(2\Delta p_x)$). In an extreme case, a “plane-wave” has a definte value of momentum $\vec{p}$ ($\Delta\vec{p} = 0$), and so its position is completely undetermined ($\Delta\vec{r} = \infty$).

- Particles which have precisely defined energies (ΔE is small) must have correspondingly long lifetimes ($\Delta t \geq \hbar/(2\Delta E)$). Only completely stable particles ($\Delta t = \infty$) can have a definite energy value.

- Short-lived particles have high uncertainties in their energies ($\Delta E \geq \hbar/(2\Delta t)$).

- Example: Imagine an electron trapped in a box of side length $2a$ ($\Delta x = a$). The corresponding uncertainty in each component of its momentum is given by $\Delta p_x = \hbar/2a$. For $a = 0.05\,\text{nm}$ (the hydrogen atom),

$$\Delta p_x = \frac{197.3\,(\text{eV}/c)\,\text{nm}}{2(0.05\,\text{nm})} = 1973\,\text{eV}/c$$

A lower bound estimate of the average kinetic energy, which can be related to this uncertainty, is

$$\langle K \rangle = \frac{\langle p_x^2 + p_y^2 + p_z^2 \rangle}{2m} = \frac{(\Delta p_x)^2 + \ldots}{2m} \geq \frac{3\hbar^2}{8ma^2}$$

For $a = 0.05\,\text{nm}$, $\langle K \rangle \geq 3(1973\,\text{eV}/c)^2/2(511000\,\text{eV}/c^2) = 11.4\,\text{eV}$. The actual kinetic energy of an electron in a hydrogen atom is $13.6\,\text{eV}$.

- The size of the hydrogen atom can be estimated by minimizing the total energy

$$K + U = \frac{3\hbar^2}{8ma^2} - \frac{e^2}{4\pi\epsilon_0 a}$$

Setting the derivative to zero gives

$$a = \frac{3}{4}\,\frac{\hbar^2 4\pi\epsilon_0}{me^2}$$

Except for the factor of $\frac{3}{4}$, this is the Bohr radius of the hydrogen atom. This ultimately explains why the electron doesn't collapse onto the proton.

- Attempts to confine the electron in a tighter space result in much higher kinetic energies. See Example 4.7 (p. 118[119]).

- Example: An electron of momentum $22.6\,\text{keV}/c\,\vec{\imath}$ ($\lambda = 54.9\,\text{pm}$) passes through a single slit of width 10 nm oriented along the y direction. Before entering the slit, $\Delta y = \infty$ (no knowledge of its y coordinate) and so Δp_y can be zero, but after leaving the slit (assuming it gets through — not all of them do), $\Delta y \approx 5\,\text{nm}$. New uncertainty in p_y is

$$\Delta p_y \geq \frac{197.3\,(\text{eV}/c)\,\text{nm}}{2(5\,\text{nm})} = 19.7\,\text{eV}/c$$

This gives rise to a angular spread of

$$\Delta\phi \approx \Delta p_y/p_x \geq 8.7 \times 10^{-4}\,\mathrm{rad}$$

Compare this with the angular spread predicted by diffraction:

$$\Delta\phi \approx \frac{\lambda}{a} = \frac{54.9\,\mathrm{pm}}{10\,\mathrm{nm}} = 5.5 \times 10^{-3}\,\mathrm{rad}$$

greater than above by a factor of 2π.

- Table of masses and lifetimes ($\hbar/2 = 0.3291\,\mathrm{eV\,fs}$):

Particle	Rest energy	Lifetime	Energy width	Borrow time
$e^{\pm}$	0.511 MeV	∞	0	0.644 zs
$\mu^{\pm}$	105.7 MeV	2.2 μs	0.15 neV	3.11 ys
$\tau^{\pm}$	1777 MeV	0.30 ps	1.1 meV	0.185 ys
$\pi^{\pm}$	140 MeV	26 ns	13 neV	2.35 ys
π^{0}	135 MeV	84 as	3.9 eV	2.44 ys
$\rho^{\pm}$	769 MeV	4.5 ys	73 MeV	0.428 ys

Note: Third and fourth columns are equal to $\hbar/2$ divided by the second and first columns, respectively.

- The "energy width" represents the uncertainty in a particle's rest energy (this would show up in the energy spectrum of a particle, either by direct measurement or inference from the decay particles). This uncertainty can be related to the lifetime of the particle. For example, a muon has a lifetime of 2.2 μs, and thus has an energy uncertainty of 0.15 neV (undetectable). In contrast, the $\rho^{\pm}$ mesons have a lifetime of 4.5 ys and an energy width of 73 MeV (about 10 % of the total rest energy). In fact, this lifetime is not measured directly, but is inferred from the energy spectrum. Fig. 4.28[26] on p. 130[133] shows the measured energy spectrum of the Δ resonance; Q4-20[17] asks a question about it.

- The energy/time uncertainty principle can also be interpreted as representing the time scale over which a certain amount of energy can be "borrowed" from the vacuum in the creation of "virtual particles".

- Example: e^-/e^+ pairs may spontaneously appear in the vacuum, borrowing their energy from "nowhere". This requires an uncertainty in the

definition of energy of at least $2m_ec^2 = 1.022\,\text{MeV}$, and thus this pair has a mean lifetime on the order of

$$\Delta t = (\hbar/2)/\Delta E = 0.322\,\text{zs}$$

before recombining.

- Example: The strong interaction between two nucleons in a nucleus can be thought of as being mediated by the exchange of a π^0 meson between the two nucleons. Creation of this virtual particle requires 135 MeV, and so the virtual particle has a mean lifetime of $\Delta t = 2.44\,\text{ys}$. This limits the range of the strong nuclear force to

$$\text{Range} = c\Delta t = (3.00 \times 10^8\,\text{m/s})(2.44 \times 10^{-24}\,\text{s}) = 0.732\,\text{fm}$$

which is on the order of the size of the nucleus. Only adjacent nucleons in a nucleus experience this force (which is why large nuclei are unstable; electrostatic repulsion is long range).